Circular Motion

Learning Objectives

By the end of this section, students will be able to:

- Define period, frequency, angular displacement, and angular velocity for circular motion
- Solve for the centripetal acceleration of an object moving on a circular path.
- Use the equations of circular motion to find the position, velocity, and acceleration of a particle executing circular motion.
- Explain the equation for centripetal acceleration.
- Apply Newton's second law to develop the equation for centripetal force in the radial direction.
- Use circular motion concepts in solving problems involving Newton's laws of motion.
- Identify inertial and non-inertial reference frames.

Circular Motion & Centripetal Force

1. Introduction to Circular Motion

We encounter circular motion everywhere around us. The moon revolves around the Earth in an almost circular orbit. When lifting dumbbells during arm curls, the forearm moves in a circular path. Cars navigating curved roads and runners running on curved tracks are also examples of circular motion.

Before we start investigating circular motion, let's make a few observations and predictions about objects performing circular motion. Figure 1 shows an object moving with a velocity $\vec{v}$ on a circle of radius R.

Think about these questions.

1. How would you describe the direction of the velocity with relation to the circular path at any point on the circle (is it along a tangent to the circle or along the radius)?
2. Does the direction of velocity change or remain the same as the object moves along the circle? Use the directions of the velocity at four locations in Figure 1 to answer this question.
3. If the direction changes, is the direction constantly changing or only at certain locations in the path?
4. Assume that the speed of the object is constant during motion. We call this special case **uniform circular motion**. Is there acceleration involved in this motion?
5. For the object in 4., if the direction of the velocity changes but the speed does not, from our chapter in kinematics, what should the direction of the acceleration be with respect to the velocity?
6. For the circular motion, which direction is perpendicular to the velocity? Think in terms of circular parameters such as tangent and radius.
7. If there is acceleration, what should be its direction?

Answers are provided below, but you should think carefully about all the concepts above.

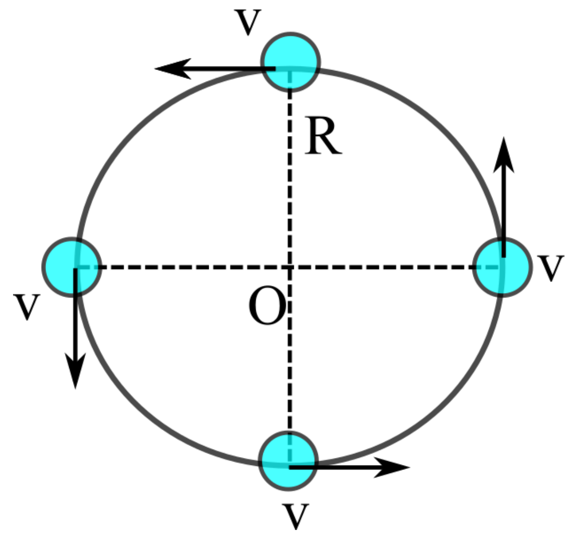

Figure 1: An object moving on a circular path of radius R

Answers

1. The velocity is along the tangent to the circle.

2. The direction of the velocity changes in space as the object moves around the circle. But in relation to the circle, it is always tangential.

3. The direction constantly changes as the object moves on the circle to point in the direction of the tangent to the circle.

4. Yes. The direction of the velocity changes even if the speed is constant. There is an acceleration responsible for the change in direction.

5. For a change in direction only, without a change in speed, the acceleration needs to be perpendicular to the velocity.

6. In a circle, the direction along the radius is always perpendicular to the tangent at any point.

7. At any point on the circular path, the acceleration should point along the radius and towards the center of the circle.

2. Circular Motion & Angular Position

Circular motion is important in the movement of the human body because most movements of bones about joints are circular. Joints are locations in the human body where bones meet, allowing movement between them. Let us start by studying the terms to describe rotational motion or **rotational kinematics**.

Studying this topic illustrates most concepts associated with rotational motion and leads to the study of many new topics that we group under the name **rotation**. Pure rotational motion occurs when points in an object move in circular paths centered on **one point** or a **fixed axis of rotation**. Pure translational motion is motion without rotation. Some types of motion combine translational as well as rotational motion, such as lifting a dumbbell with the forearm, and a rotating hockey puck moving along ice.

2.1. Angular Position

To describe the angular position of an object, we need a **line of reference**. Once this line is identified, all angles are measured with respect to this line (see Figure 2).

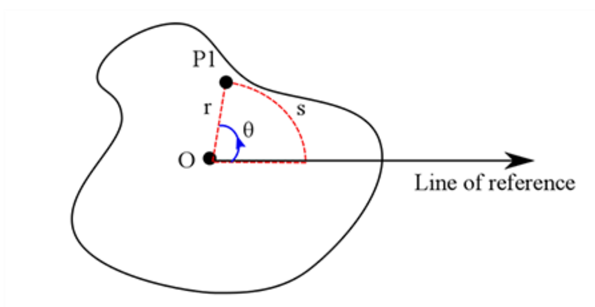

Figure 2: Angular position

Consider point P1 at a distance r from O. We can measure the angle, θ, of P1 with respect to the line of reference. θ is known as the **angular position** of P1. We follow a convention such that, for **counterclockwise** rotation, θ is **positive;** for **clockwise** rotation, θ is **negative.**

This line of reference could be fixed in space. Angles measured in this way are called absolute angles. The line of reference could also move in space. Angles measured with respect to moving lines of reference are called relative angles. For example, you can rotate your forearm about the elbow (see Figure 3) and measure the angle with respect to the horizontal (absolute angle). This measurement θ_A is an example of an absolute angle. You could also describe the rotation by using the arm (the arm can move in space) as the line of reference, measuring the angle between your arm and the forearm (relative angle). The angle θ_R in Figure 3 is an example of a relative angle.

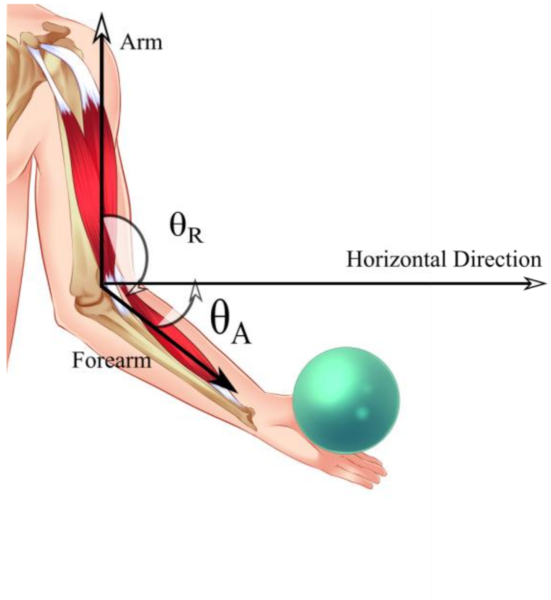

Figure 3: When the forearm rotates, the angle can be measured with respect to a fixed line of reference (e.g., the horizontal direction) - this is an example of absolute angle. The angle can also be measured with respect to the movable arm - this is an example of a relative angle.

Angular position is measured with an instrument known as the **goniometer,** as shown in Figure 4. Digital goniometers are also available. Physical therapists use goniometers to measure the **range of motion** about joints. This measurement is useful for the physical therapist to compare the benefits of treatment before and after intervention.

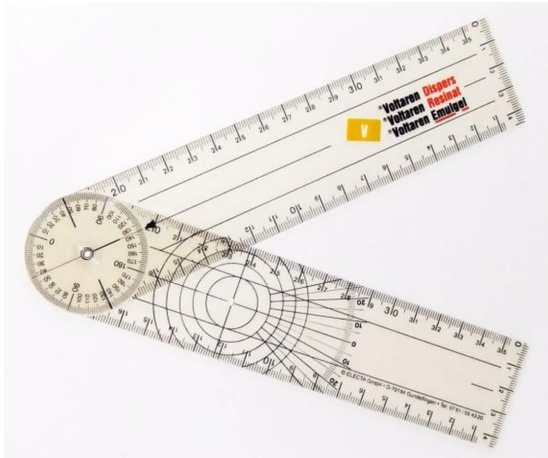

Figure 4: A goniometer is used to measure angles. Image from http://commons.wikimedia.org/wiki/File:Medizinischer_Goniometer.jpeg

2.2 Arc length (s)

Arc length is the distance traveled by an object when moving around a circle in a given time. From geometry, we know that the arc length, s, in Figure 2, is related to the radius and the angle covered by the arc by:

$$s = r\theta$$

$$\therefore \theta = s/r$$

The unit of θ is **radian or rad**. A radian is a dimensionless quantity.

One radian is defined as the angle P1 subtends with O after traveling a distance 1 r along the circle.

2.3 Converting between degrees and radians

To convert from degrees to radians and vice versa, it is important to note that in 1 revolution, $360°$ are covered.

The distance covered by the object in one revolution is the circumference of the circle $s = 2\pi r$, where r is the radius of the circle.

Therefore, in one revolution, $\theta = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi\ rad$

$$1\ revolution \equiv 360° \equiv 2\pi\ rad$$

$$1 \text{ radian} = 360°/2\pi\ = 57.3°$$

For more than one revolution θ can be calculated as follows:

$$\theta \text{ after 2 revolutions} = 720° = 4\pi \text{ rad}$$

$$\theta \text{ after } n \text{ } r\text{evolutions} = 360° \times n = 2\pi n \text{ rad}$$

Self-Assess

What is the relationship between the arc length Δs and the rotational angle Δθ traversed by an object moving in a circle of radius r? What are the units of Δθ measured using this relationship?

Answer: $\Delta\theta = \frac{\Delta s}{r}$.

$\Delta\theta$ is measured in radians – a dimensionless quantity.

2.3 Angular Displacement, Angular Velocity, Frequency, and Period

We now define quantities similar to displacement and velocity in translation motion for rotational motion, such as **angular displacement** and **angular velocity.**

(a) Angular Displacement $\Delta\theta$

Angular Displacement: Angular displacement $\Delta\theta$ is the **change in angular position**.

$$\Delta\theta = \theta_f - \theta_i$$

Where θ_f: The final angular position of the object

θ_i: The initial angular position of the object

For counterclockwise rotation $\Delta\theta$ is positive.

For clockwise rotation, $\Delta\theta$ is negative.

(b) Angular Velocity ω

Angular Velocity: Angular velocity ω is the rate of **change of angular displacement** with time.

$$\omega_{avg} = \frac{\theta_f - \theta_i}{t_f - t_i} = \frac{\Delta\theta}{\Delta t}$$

ω can be **clockwise or counterclockwise**. Units of ω are rad/s.

The angular velocity ω is a vector. Its direction can be found using the **right-hand rule**.

Right-Hand Rule: Curl the fingers of your right hand in the direction of the rotation; then the direction of the outstretched thumb gives the direction of ω. Figure 5 shows that for counterclockwise rotation, the thumb of the right hand points in the positive z direction. Therefore, ω points in the positive z-direction.

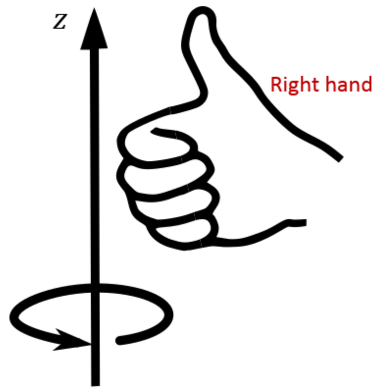

Figure 5: Direction of angular velocity for counterclockwise rotation

Figure 6 shows that for clockwise rotation, the thumb of the right hand points in the negative z direction.

Therefore, ω points in the negative z-direction.

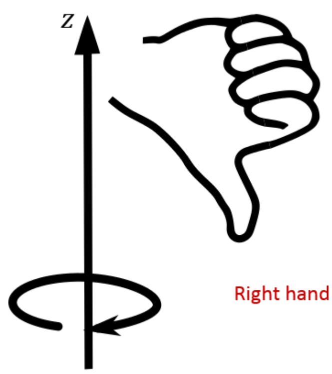

Figure 6: Direction of angular velocity for clockwise rotation

2.4 Relation Between Angular Speed, Frequency of Revolution, and Period of Revolution

Frequency: Frequency (f) is defined as the number of revolutions completed by the object per second. The SI units of frequency are revolutions per second. The angular velocity, sometimes known as angular frequency, is related to f as follows

$$\omega = 2\pi f$$

Period of revolution/rotation (T): The period of revolution is the time required to finish one complete rotation or revolution about the axis.

$$T = \frac{1}{f} = \frac{2\pi}{\omega}$$

2.5 Relationship Between Linear Velocity and Angular Velocity

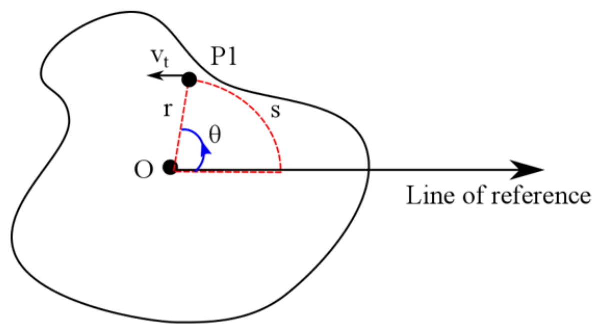

Figure 7: Cross-section of an object rotating about an axis passing through O and perpendicular to the screen.

Consider particle P1 rotating as shown in Figure 7. If it starts at $\theta = 0$, then the distance covered in time t is the arc-length s.

$$s = r\theta$$

Average linear speed (Also known as tangential speed) is given by

$$v_t = \frac{s}{t} = \frac{r\theta}{t} = r\omega$$

In terms of period: $v_t = \frac{2\pi}{T}r$

In kinesiology, knowing typical angular velocities of joints helps evaluate the skills of patients and athletes during various anatomical movements to plan suitable exercises for treatment or performance improvement.

Exercise: Try measuring the angular velocity for these movements. Compare your answers with your classmates.

1. **Angular velocity of knee extension during sit-to-stand**: What is the angular displacement of the thigh in going from the sitting position to the standing position?

Measure the time taken for you to stand from the sitting position. Repeat a few times and take the average of the times.
Calculate the average angular velocity in deg/s.

2. **Angular velocity of elbow flexion during an arm curl**: What is the angular displacement of the forearm at the elbow joint during an arm curl, with the forearm being vertically down to the forearm being vertically up? Measure the time taken by you for this flexion. Repeat a few times and take the average.
Calculate the average angular velocity in deg/s.

Try measuring these other angular velocities: knee extension during kicking a soccer ball and shoulder flexion in throwing a ball.

Self-Assess

What is the angular velocity of a particle moving in a circle of radius r? How is it related to the tangential velocity of the particle? What directions are associated with linear velocity?

Answer: Angular velocity is defined as the rate of change of rotational displacement.

The relationship between the angular velocity and the tangential velocity is

$$v = r\omega$$

Angular velocity has only two directions with respect to the axis of rotation—it is either clockwise or counterclockwise.

3. Acceleration in Circular Motion

We know from kinematics that acceleration is a change in velocity, either in its magnitude or in its direction, or both. In uniform circular motion, the direction of the velocity changes constantly, so there is always an associated acceleration, even though the magnitude of the velocity might be constant. In this section, we examine the direction and magnitude of that acceleration.

Figure 8 shows an object moving in a circular path at constant speed. The direction of the instantaneous velocity is shown at two points along the path (tangent to the circle). Acceleration is in the direction of the change in velocity, which points directly toward the center of rotation (the center of the circular path). This pointing is shown with the vector diagram in the figure. We call the acceleration of an object moving in uniform circular motion (resulting from a net external force) the centripetal acceleration (a_c); centripetal means "toward the center" or "center seeking." If v_1 and v_2 are very close on the circle, then Δv points along the radius of the circle.

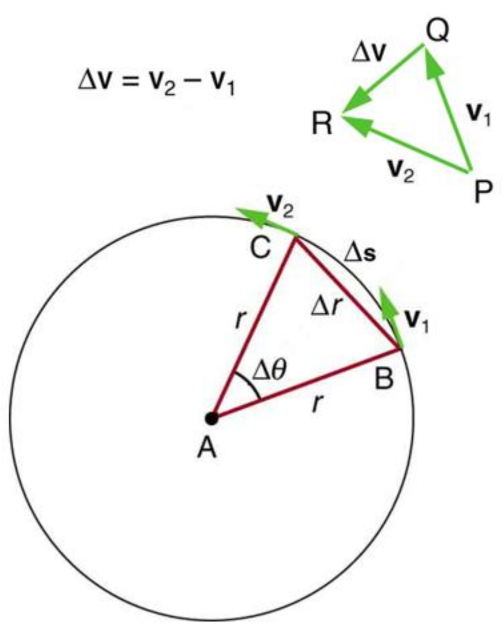

Figure 8: An object moving in a circular path at constant speed. While the magnitude of the velocity does not change, its direction does.

The direction of centripetal acceleration is along the radius of the circle towards the center.

The direction of centripetal acceleration is toward the center of the curvature, but what is its magnitude? Note that the triangles formed by the velocity vectors, PQR, and the one formed by the radii r and Δs, ABC, are similar because both the triangles ABC and PQR are isosceles triangles and $\angle BAC = \angle QPR$. The two equal sides of PQR are the speeds $v_1 = v_2 = v$. Using the properties of two similar triangles, we obtain

$$\frac{\Delta v}{v} = \frac{\Delta s}{r}$$

The magnitude of acceleration is $\frac{|\Delta \vec{v}|}{\Delta t}$, and so we first solve this expression for $|\Delta \vec{v}|$ or simply Δv:

$$\Delta v = \frac{v\Delta s}{r}$$

Then we divide this by Δt, yielding $\frac{\Delta v}{\Delta t} = \frac{v\Delta s}{r\Delta t}$

Finally, noting that $a_c = \frac{\Delta v}{\Delta t}$ and $v = \frac{\Delta s}{\Delta t}$, the magnitude of the centripetal acceleration is

$$a_c = \frac{v^2}{r} = r\omega^2$$

which is the acceleration of an object in a circle of radius r at a speed v.

3.1 Centripetal Acceleration

The acceleration needed for an object moving in a circle of radius R with constant speed v is known as the **centripetal or radial acceleration**.

The magnitude of the centripetal acceleration is $a_c = \frac{v^2}{R}$.

The direction of a_c is towards the center of the circle along the radius.

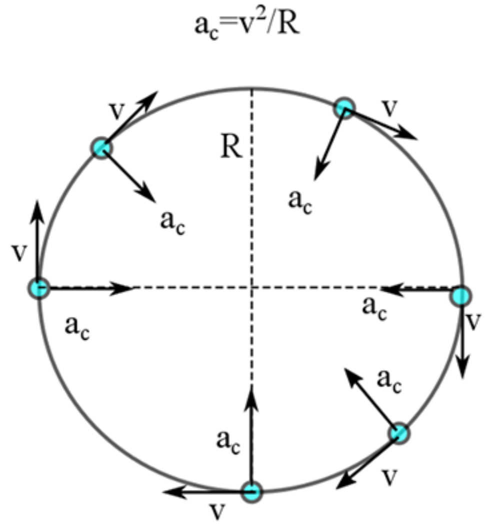

Figure 9: Velocity and centripetal acceleration in uniform circular motion

Self-Assess

For a particle moving in a circle with constant speed *v*, what is the direction of the centripetal acceleration? What is the angle between the particle's velocity and the acceleration? What role does centripetal acceleration play in the particle's circular path?

Answer: For a particle moving in a circle with constant speed v, the centripetal acceleration acts along the radius of the circle, pointing towards the center of the circle. The particle's velocity and acceleration are perpendicular to each other. The centripetal acceleration is responsible for changing the direction of the particle's velocity while keeping the speed constant.

3.2 Simulation: Centripetal Acceleration

The simulation in the link below helps you visually observe that the centripetal acceleration points along the radius and towards the center of the circle.

Centripetal Acceleration – GeoGebra (License CC-BY-SA, GeoGebra Terms of Use)

Activity:

In this simulation, you are going to observe the motion of the particle between the positions x_1 and x_2. You can select these positions using the slider.

The magnitude of the velocity is constant during this motion and can also be selected using the slider.

The direction of acceleration will be deduced by using the formula $\vec{a}_{vec} = \frac{\Delta\vec{v}}{\Delta t}$, where

$\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$.

1. Select x_1 and x_2, the magnitude of the velocity and the radius. Start the simulation.
2. As the particle moves, the direction of $\vec{v}_2$ and $\vec{v}_1$ change. The triangle in the right-side plots $\Delta\vec{v} = \vec{v}_2 - \vec{v}_1$ using graphical vector subtraction.
3. The acceleration points in the same direction as the vector $\Delta\vec{v}$. The vector $\Delta\vec{v}$ is also shown on the circle to get a sense of where it lies in relation to the circle. As the change $\Delta\vec{v}$ takes over the time interval Δt, its location is shown between the two velocities.

Centripetal acceleration is an instantaneous acceleration:

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{v}}{\Delta t}$$

As $\overrightarrow{v_1}$ moves towards $\overrightarrow{v_2}$, observe the direction of $\Delta\vec{v}$. What happens when the velocities are really close? In what direction does $\Delta\vec{v}$ points as the particle moves for a very small interval of time?

4. Centripetal Force

We know that an object moving in a circular path has an acceleration equal to the centripetal acceleration. What is the source of this acceleration?

Any force or combination of forces acting **along the radius** of a circle can cause a centripetal or radial acceleration.

For an object of mass m moving in a circle of radius r with a constant speed of v, a real force or a combination of real forces must be present. Its magnitude should equal $ma_c = \frac{mv^2}{r}$ and it should be directed towards the center of the circle along the radius.

This net force is called the centripetal force F_c. A few examples of centripetal forces are the tension in the rope on a tether ball, the force of Earth's gravity on the Moon, the friction between roller skates and a rink floor, a banked roadway's force on a car, and forces on the tube of a spinning centrifuge.

4.1 Newton's Second Law for Circular Motion

Let us apply Newton's Second Law along the radial direction in a circle. Figure 10 shows a mass *m* moving with speed *v* around a circle with center O and radius R. The radial direction is indicated by the dotted line. We choose the direction pointing **towards the center O as positive** and **away from the center as negative**. Note that the radial direction in space changes as the object moves along the circle.

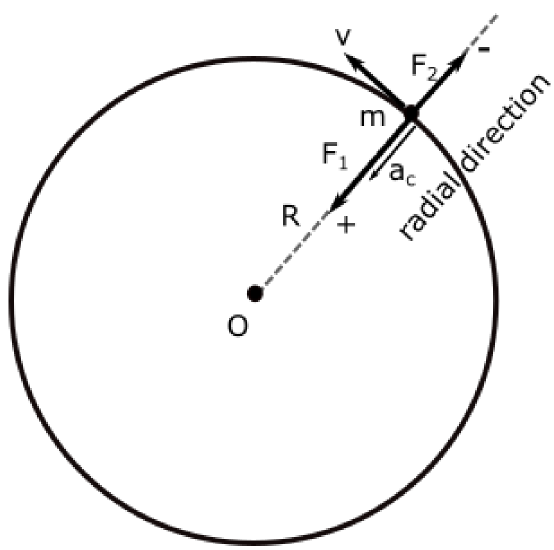

Figure 10: Circle with center O and radius R.

Newton's 2nd Law of motion along the radius for circular motion is

$$F_c = \Sigma F_r = ma_c = \frac{mv^2}{R},$$

Where ΣF_r is the sum of all the forces acting along the radius (towards or away from the center of the circle) and is known as the **centripetal force** F_c.

For circular motion to be possible:

$F_c =$ Sum of forces along **radius directed towards** the center – Sum of forces along radius directed **away from** the center.

That is, the sum of forces along radius directed towards the center > the sum of forces along radius directed away from the center.

In other words, for circular motion to be possible, there should be an unbalanced force acting along the radius towards the center of the circle.

In Figure 10,

$$\sum F_r = F_1 - F_2$$

For circular motion to be possible $F_1 > F_2$.

Self-Assess

What is the centripetal force?

Answer: Any net force causing uniform circular motion is called a centripetal force. The direction of a centripetal force is toward the center of curvature, the same as the direction of centripetal acceleration. According to Newton's Second Law of motion, net force is mass times acceleration: $\sum F_r = ma$.

For uniform circular motion, the acceleration is the centripetal acceleration given by

$a = ac = v^2/r$, where v is the speed of the particle and r is the radius.

4.2 Examples of Uniform Circular Motion

In this section, we will discuss some examples of objects moving on circular trajectories and learn how to apply Newton's Second Law for these scenarios.

4.2.1 An object tied to a string and whirled around

Consider an object of mass m tied at the end of a string of length R whirled around in a circle.

The tension in the string is T. T acts along the radius of the circle, providing the centripetal force.

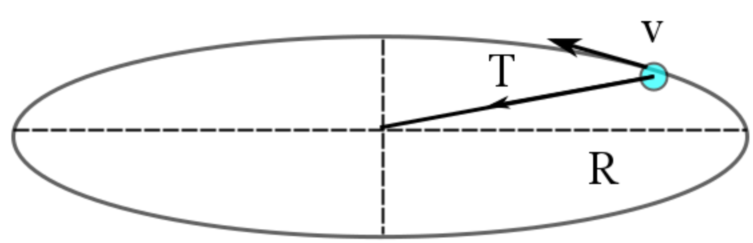

Figure 11: Object tied to the end of the string and rotated

Applying Newton's Second Law:

$$\Sigma F_r = \frac{mv^2}{R}$$

$$LHS = \Sigma F_r = T$$

$$T = \frac{mv^2}{R}$$

Note: T here is tension, not the period of revolution

4.2.2 Moon orbiting the Earth

The moon revolves around the Earth in an almost circular orbit of radius R, shown in Figure 12.

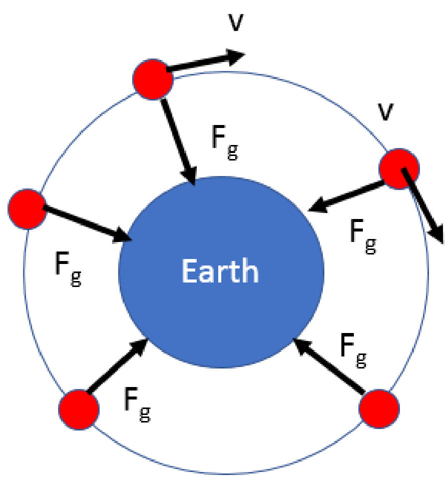

Figure 12: The moon revolving around the Earth.

The gravitational force between any two masses m and M separated by a distance r is **attractive** and acts along the **line joining the two masses**.

The magnitude is given by

$F_g = \frac{GMm}{r^2}$ where $G = 6.67 \times 10^{-11} \frac{Nm^2}{kg^2}$

Applying Newton's Second Law in the radial direction

$$\Sigma F_r = \frac{mv^2}{r}$$

The gravitational force provides the centripetal acceleration.

$$LHS = \Sigma F_r = \frac{GM_E m}{r^2}$$

$$\frac{GM_E m}{r^2} = \frac{mv^2}{r}$$

Solve for v: $v = \sqrt{\frac{GM_E}{r}}$

Self-Assess

The moon revolves around the Earth in a circular orbit. What provides the centripetal force?

Answer: The force of gravity between the Earth and the Moon provides the centripetal force.

4.2.3 A car negotiating a curve on the road

Figure 13 shows the top view of a car negotiating a curve. The force of static friction provides the centripetal force. A large radius implies a small centripetal force for a given speed. A small friction force is needed to sustain the motion. A smaller radius implies a larger centripetal force for the same speed. A large friction force is needed to sustain the motion.

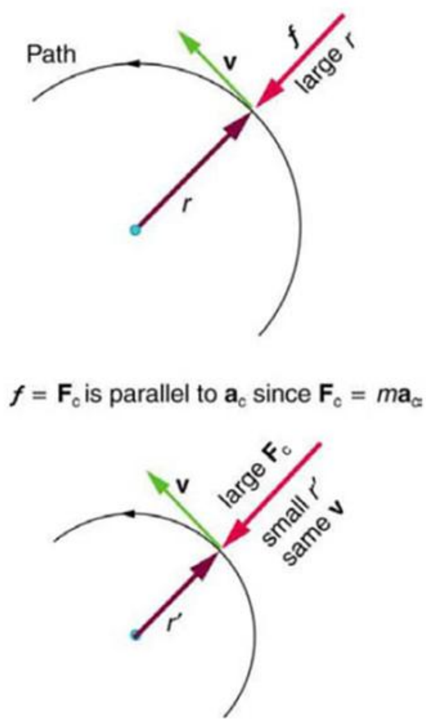

Figure 13: Car negotiating a curve

Figure 14 shows the back view of a car negotiating a curve with constant speed v. The radial direction is the horizontal direction pointing to the left. Three forces act on the car: the weight w, the normal force N, and the friction force f. The only force in the radial direction is f. Note that the friction force is static friction because there is no motion in the radial direction.

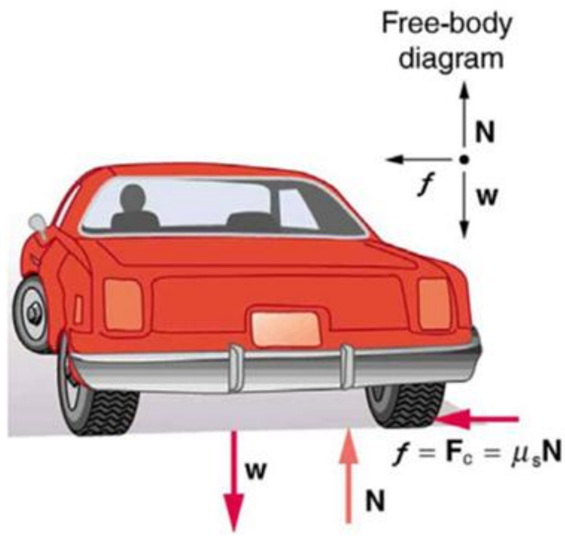

Figure 14: Back view of a car negotiating a curve

Applying Newton's Second Law in the radial direction

$$\Sigma F_r = \frac{mv^2}{r}$$

$$LHS = \Sigma F_r = f$$

$$f = \frac{mv^2}{r}$$

$$f = force\ of\ static\ friction$$

As the speed of the car increases, the force of static friction also increases to its maximum value.

To find the maximum velocity on a curved road a car can have without skidding, we use the maximum force of static friction f_{max}

$$f_{max} = \frac{m{v_{max}}^2}{r}$$

But $f_{max} = \mu_s N$.

Now applying Newton's Second Law in the y-direction,

$$\Sigma F_y = N - mg = 0 \Rightarrow N = mg$$

Using this value in the equation below,

$$f_{max} = \frac{mv_{max}^2}{r}$$

$$\mu_s N = \mu_s mg = \frac{mv_{max}^2}{r}$$

Solving for v_{max}, $v_{max} = \sqrt{\mu_s gr}$

Observations about the maximum velocity on a curved road:

$$v_{max} = \sqrt{\mu_s gr}$$

1. The maximum velocity does not depend on the mass of the vehicle. It is the same for a small car or a large truck.
2. The coefficient of static friction, μ_s, changes with road conditions. For example, on a rainy day μ_s might decrease substantially, making the road conditions hazardous to drive.

4.2.4 Car on a Banked Road

Sometimes the friction provided by the road is not enough to sustain the motion of cars along a curved road.

One solution is to bank the road – i.e., raise one side of the road with respect to the other through an angle θ (angle of banking).

The normal force on the car is now perpendicular to the banked road. When resolved in the horizontal and vertical directions, the component of N in the horizontal direction $N\sin\theta$ acts along the radius of the circular path. $N\sin\theta$ provides the centripetal force.

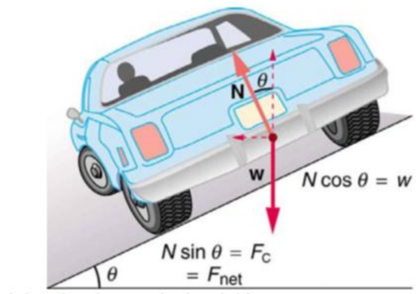

Figure 15: Car on a banked road

Applying Newton's Second Law in the radial direction

$$\Sigma F_r = \frac{mv^2}{r}$$

$$LHS = \Sigma F_r = N \sin\theta$$

$$N \sin\theta = \frac{mv^2}{r}$$

To find the angle of banking, we know:

$$N \sin\theta = \frac{mv^2}{r} \quad (1)$$

To find N, we use Newton's Law in the y direction:

$$\Sigma F_y = ma_y = 0$$

$$N \cos\theta - mg = 0 \Rightarrow N = \frac{mg}{\cos\theta} \quad (2)$$

Substituting (2) in (1)

$$\frac{mg}{\cos\theta} \sin\theta = \frac{mv^2}{r} \Rightarrow \tan\theta = \frac{v^2}{rg}$$

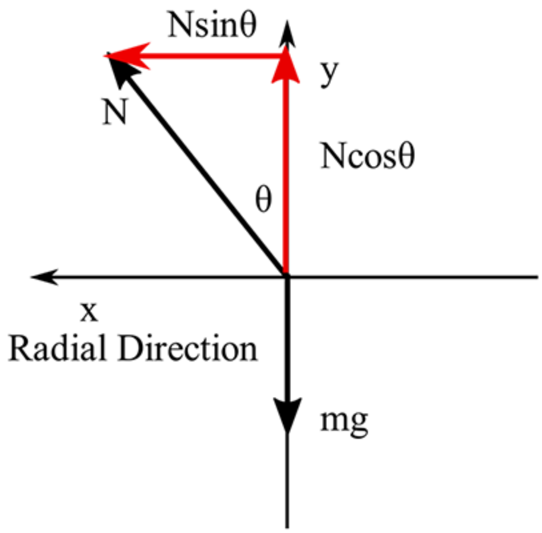

Figure 16: Free body diagram of the car on a banked road

Self-Assess

On a banked road, what force provides the centripetal force?

Answer: On a road banked at an angle θ, the component of the normal force, N, in the direction of the radius of the curved road (Nsinθ), provides the centripetal force.

4.2.5 Centripetal force provided by a combination of forces

Consider a car moving on a circular hill of radius r.

On the very top of the hill, the forces acting on the car are its weight acting radially down, and the normal force acting radially up.

Net force in the radial direction = $mg - N$, which provides the centripetal force for the car to navigate the hill.

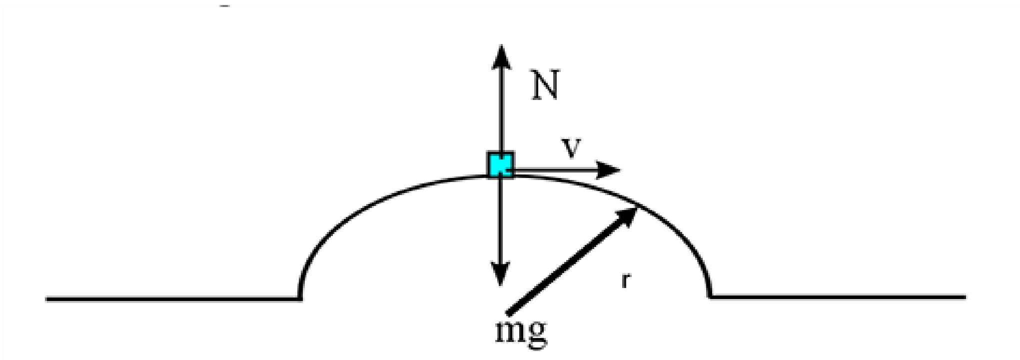

Figure 17: Car on top of a circular hill of radius r

Let the vertical direction be positive. When the car is on the top of the hill, the vertical direction is the radial direction.

Applying Newton's Second Law in the radial direction

$$\Sigma F_r = \frac{mv^2}{r}$$

$$LHS = \Sigma F_r = -N + mg$$

$$mg - N = \frac{mv^2}{r}$$

From the equation above, we see that the weight has to be greater than the normal force ($mg > N$) for a centripetal force to exist. When $mg \leq N$, the car cannot be maintained on the curved road and will skid off and lose contact.

4.2.6 Runner on a Circular Track

You may have noticed that when runners are running on a circular or curved track, they lean to one side. Can you explain that? The runner in **Error! Reference source not found.** is running on a circular track with radius R with a speed v. The radial direction is labeled as *r*, and the vertical direction is labeled as *v*. As the runner puts her foot on the ground the ground the ground reaction force is shown as F_G. **Error! Reference source not found.** also shows the radial and vertical components as dotted arrows.

To maintain this speed, should the runner lean? If so, what should the leaning angle be?

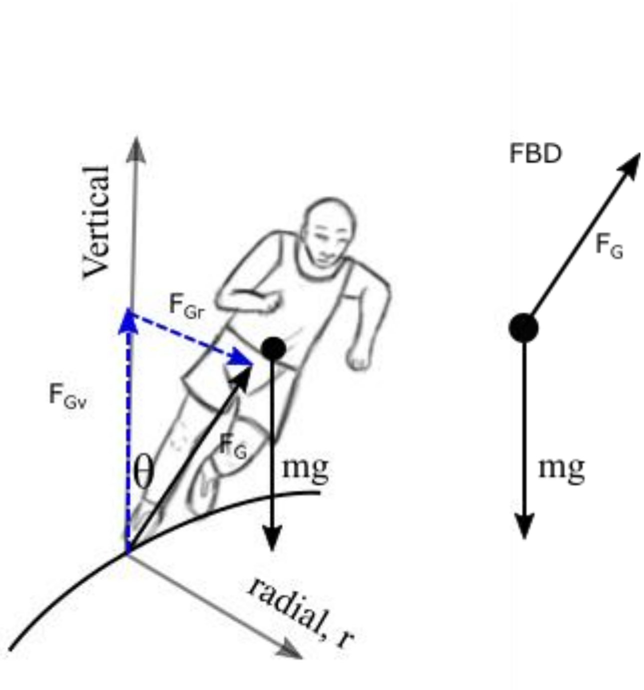

Figure 18: Runner on a circular track of radius R, leaning with respect to the vertical

1. For the runner to be on the circular track, in which direction does she need a force? Study **Error! Reference source not found.** to answer.

Answer: The runner needs a centripetal force in the radial direction shown in the figure.

2. Is there a force in that direction? Study **Error! Reference source not found.** to answer.

Answer: We can resolve the ground reaction force in the radial direction and the vertical direction. The component of the force in the **radial direction** provides the centripetal force.

3. Is there a force in that direction?

Answer: We can resolve the ground reaction force in the radial direction and the vertical direction. The component of the force in the radial direction provides the centripetal force.

4. Can you express the radial and vertical components of the ground reaction force in terms of the angle θ? Use the right triangle in **Error! Reference source not found.** to answer.

Answer:

$$F_{Gr} = F_G \sin\theta$$

$$F_{Gv} = F_G \cos\theta$$

Set up Newton's Laws equations in the vertical and radial direction.

5. Vertical direction: Is there an acceleration?

Answer: No.

Newton's Second Law in the vertical direction is

$$\Sigma F_v = F_{Gv} - mg = 0$$

$$F_{Gv} = mg$$

$$F_G \sin\theta = mg \quad (3)$$

6. Radial direction: Is there an acceleration?

Answer: Yes. There is a centripetal acceleration in the radial direction with a magnitude $\frac{v^2}{R}$

Newton's Second Law in the radial direction is

$$\Sigma F_r = \frac{mv^2}{R}$$

$$F_{Gr} = \frac{mv^2}{R}$$

$$F_G \cos\theta = \frac{mv^2}{R} \quad (4)$$

(3) divided by (4):

$$\frac{F_G \sin\theta}{F_G \cos\theta} = \frac{\frac{mv^2}{R}}{mg}$$

$$\tan\theta = \frac{v^2}{Rg} \quad (5)$$

Equation (5) gives the leaning angle of the runner to maintain a speed v on a circular track of radius R.

Study Equation (5) and answer the following questions.

1. For a given speed, does the angle depend on the mass of the runner? Does it depend on the radius?
2. If you want to run faster, should you lean less or more?
3. If you don’t lean, can you negotiate the circular track?

Click below for the answers.

1. Equation (5) is independent of the mass of the runner but depends on the radius of the track. The larger the radius, the smaller the angle the runner needs to lean from the vertical.
2. From Equation (5), if you want to run faster, you need to lean more.
3. If you don't lean, there is no centripetal force needed to be on the circular track.

5. Summary of Uniform Circular Motion

Uniform Circular Motion refers to a particle moving around a circle with constant speed v. An object undergoing uniform circular motion (UCM) has the following characteristics:

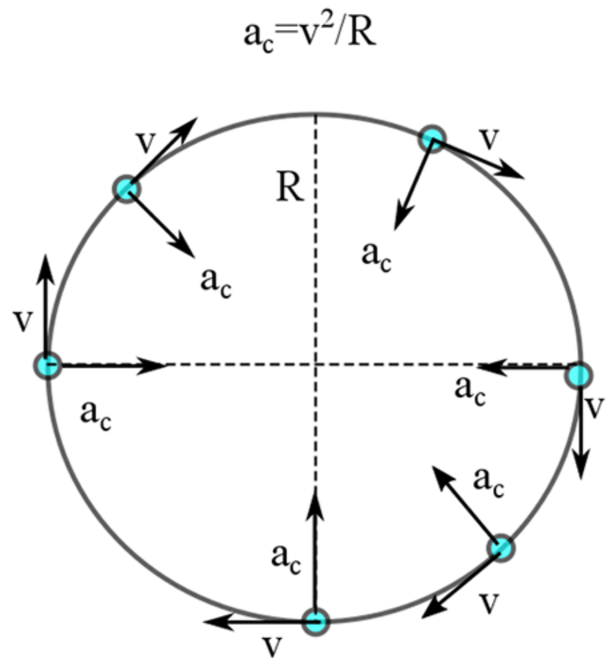

Figure 19: Particle moving in a circle

1. Velocity is tangent to the circle
2. The centripetal acceleration a_c acts along the radius towards the center of the circle.
3. The period for one revolution, T, is related to the speed v as
$$v = \frac{2\pi R}{T}.$$
4. The frequency of revolution (revolutions per s) $f = \frac{1}{T}$ are related to the speed v as $v = 2\pi Rf$.
5. Newton's 2nd Law of Motion for circular motion is
$$\Sigma F_r = ma_c = \frac{mv^2}{R}$$
Where ΣF_r is the sum of all the forces acting along the radius (towards and away from the center of the circle)

6. Inertial and Non-Inertial Frames

What happens to the driver of a car that suddenly accelerates forward?

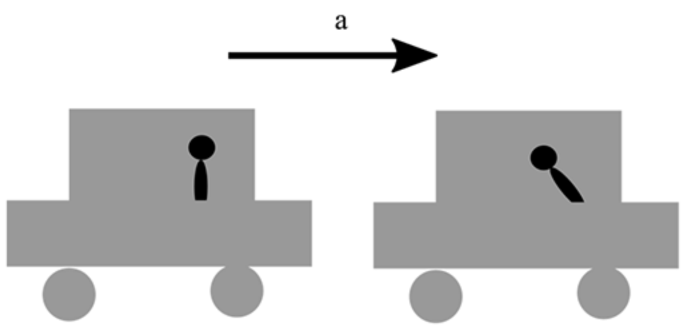

Figure 20: Accelerating car

The driver's head gets thrown backwards. What force causes this?

There is no real force causing the head to accelerate backward. The driver feels as though they are being pushed by a backward force. The backward force seems real to the driver.
This force is fictitious and has no physical origin.
The head is thrown backwards because of inertia. While the part of the body touching the seat accelerates forward, the head continues its original state of motion and seems to move backward.

For the driver to explain the backward acceleration of his head using Newton's Laws, he or she needs the fictitious force. This gives rise to the contradiction that acceleration is caused by real forces. The source of this contradiction was that the driver was trying to apply Newton's Laws in an accelerating system. **Newton's Laws are not applicable in accelerating systems.**

An accelerating car is an example of a non-inertial frame.

If a set of coordinate axes is attached to the accelerating car, Newton's Laws in this **coordinate system are not valid.**

All accelerating frames are non-inertial frames.

Some examples of non-inertial frames are accelerating cars, braking cars, turning cars, merry-go-rounds, and rotating systems.

6.1 Inertial Frame of Reference

When a passenger on the train throws a ball upwards. She observes that the ball goes straight up and then straight down. The passenger can use Newton's Laws to describe the motion of the ball and correctly predict its straight path.

Inertial frames are coordinate systems attached to frames moving at constant velocity.

Newton's Laws are valid only in inertial frames.

Some examples of inertial frames are an observer on a train moving with constant velocity and an observer on a platform.

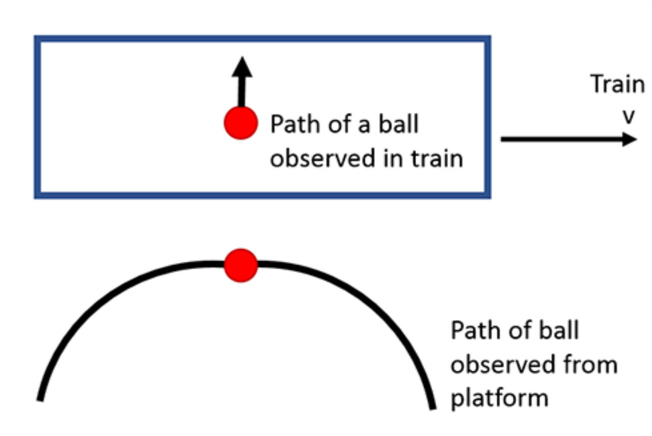

Figure 21: Inertial frames of reference

An observer on the platform observes the same ball not only going upwards but also sideways because of the train's velocity. She observes that the ball takes a parabolic trajectory. The observer on the platform can use Newton's Laws to describe the motion of the ball and correctly predict the parabolic path.

6.2 Centrifugal Force: Fictitious Force

Let us now take on a merry-go-round—specifically, a rapidly rotating playground merry-go-round. You take the merry-go-round to be your frame of reference because you rotate together.

In that non-inertial frame, you feel a fictitious force, named **centrifugal force** (not to be confused with centripetal force), trying to throw you off. You must hang on tightly to the pole. There is no force trying to throw you off. Rather, you must **hang on** to make yourself go in a circle because, otherwise, you would go in a straight line, right off the merry-go-round, because of inertia.

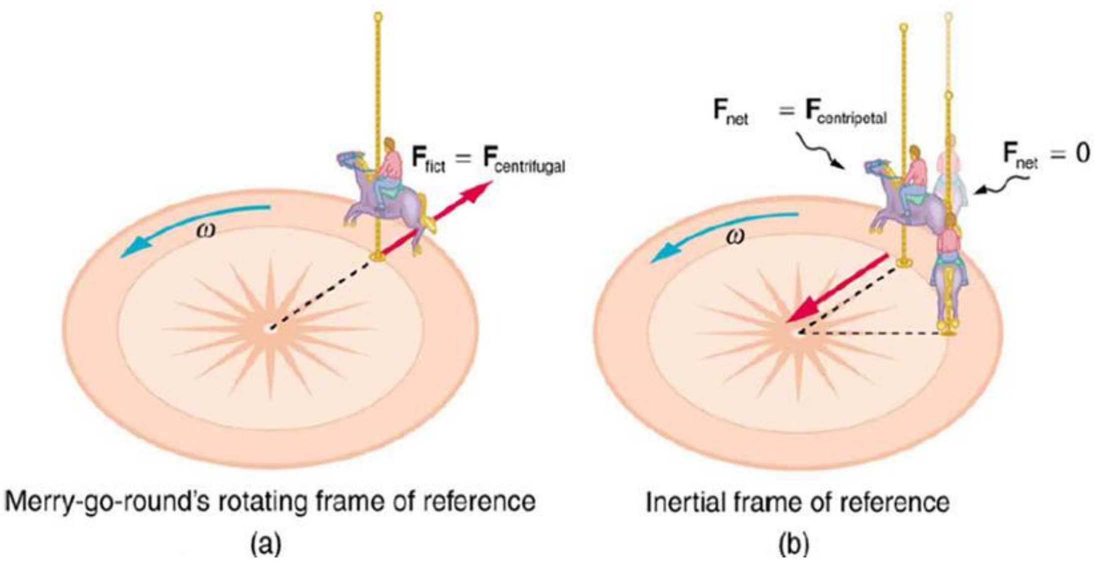

Figure 22: Centrifugal force

Self-Assess

Question: What are inertial and non-inertial frames of reference? Give examples of each.

Answer: Inertial frames are coordinate systems attached to frames moving at constant velocity. Newton's Laws are valid only in inertial frames. Some examples of inertial frames are coordinate systems attached to a train moving with constant velocity and a coordinate system attached to the platform.

Non-inertial frames are coordinate systems attached to accelerating systems. Newton's Laws are not valid in non-inertial frames. Examples of non-inertial frames are accelerating cars, braking cars, turning cars, merry-go-rounds, and rotating systems.